Three-point correlations for quantum star graphs

Marie-Line Chabanol

Institut de Mathématiques de Bordeaux, UMR 5251 CNRS-Université Bordeaux1, France

E-mail: Marie-Line.Chabanol@math.u-bordeaux1.fr

Abstract

We compute the three-point correlation function for the eigenvalues of the Laplacian on quantum star graphs in the limit where the number of edges tends to infinity. This extends a work by Berkolaiko and Keating, where they get the two-point correlation function and show that it follows neither Poisson, nor random matrix statistics. It makes use of the trace formula and combinatorial analysis.

PACS numbers: 05.45.Mt, 03.65.Sq

1. Introduction

The study of the Laplacian on a metric graph, a concept known as *quantum graphs*, now serves as a toy model for quantum chaos [1–3]. Indeed, there exists an exact trace formula relating eigenvalues and periodic orbits [1, 2, 4]. Moreover, depending on the graphs, exact computations of these orbits may be possible, whereas they are out of reach in most dynamical systems. It has thus been shown [1, 5, 6] that spectral statistics of simple generic graphs follow random matrix statistics when the size of the graph tends to infinity, as expected for chaotic quantum systems. Star graphs (graphs formed by a central vertex connected to v other vertices by edges of different lengths) play a special role because of the high degeneracy of their periodic orbits. As could be expected, this degeneracy breaks the random matrix statistics; this has been shown by the computation of the two-point correlation function [7, 8]. Moreover, it seems reasonable to expect that random matrix statistics would be retrieved by gluing some star graphs together (just by one edge). Hence star graphs can really be used as a toy model for what degeneracy can induce on statistics, and how degeneracy can be broken. Their simplicity makes exact results easier to obtain: the trace formula for star graphs has been shown to converge under quite reasonable assumptions [9]. Moreover, star graphs may also be considered as a discrete version of Seba billiards [10]; indeed the eigenvalues of quantum star graphs and the energy levels of Seba billiards are solutions of similar equations, so that their study can also say something about continuous dynamics, not only discrete one. As a step

further in the understanding of this model, we will here compute the three-point correlation function of such graphs. Moreover, it is likely to be a useful ingredient for the computation of the two-point correlation function for glued star graphs. Gluing decreases degeneracy, so that one could expect another intermediate statistics (double star graphs have been considered in [11], but their graphs are Fourier graphs, not Neumann graphs as considered here).

Spectral statistics for quantum graphs and the trace formula relating them to periodic orbits will be recalled in the first part. The second part will state the two-point correlation function as obtained in [8]. The third part will present the computation of the three-point correlation function. Perspectives regarding glued graphs will be given in the conclusion.

2. Quantum graphs: eigenvalues and trace formula

We will start by some vocabulary and notations. Let $G = (E, V)$ be a graph with a metric structure: to each edge $(i, j) \in E \subset V \times V$ is assigned a length l_{ij}, such that $l_{ij} = l_{ji}$; although the graph is supposed to be non-oriented, that is $(i, j) \in E \Rightarrow (j, i) \in E$, and $l_{ij} = l_{ji}$, we will consider the edges to be oriented: (i, j) is different from (j, i), it really describes the edge going from i to j. On each edge (i, j), one can thus define a coordinate x such that $x = 0$ corresponds to the vertex i, and $x = l_{ij}$ corresponds to the vertex j. A *periodic orbit of length n* is a set of n edges $(p_1, \ldots, p_n)$ such that p_i ends where p_{i+1} starts (as well as p_n and p_1). A periodic orbit is called *primitive* if it is not the repetition of a shorter periodic orbit. A primitive orbit repeated r times is a non-primitive orbit with *repetition number r*. We will denote by v_j the *valence* of the vertex j, that is the number of its neighbors. On each edge (i, j), one looks for the spectrum of the Laplacian. In other words, one wants to find λ and ψ_{ij} such that $-\frac{d^2 \psi_{ij}}{dx^2} = \lambda^2 \psi_{ij}(x)$. As one looks for eigenfunctions defined on the whole graph, one imposes continuity relations at each vertex, $\psi_{ij}(0) = \psi_{ik}(0)$. Moreover, the function should have a unique value on a given point, regardless of the sense of the edge it belongs to; hence, one wants $\psi_{ij}(x) = \psi_{ji}(l_{ij} - x)$. Finally, one imposes Neumann condition on each vertex $\sum_j \frac{d\psi_{ij}}{dx}|_0 = 0$. It is then a simple exercise to check that the eigenvalues λ are the solutions of $\det(I - e^{-i\lambda L} S) = 0$, where S and L are $|E| \times |E|$ matrices: L is diagonal with the length of each edge as a diagonal element, and S is defined by $S_{(i,j),(j,k)} = -\delta_{i,k} + \frac{2}{v_j}$.

The trace formula as obtained in [1] (a derivation specific to star graphs is given in [9]) states that if $d(\lambda) = \sum_n \delta(\lambda - \lambda_n)$ is the spectral density, then $d(\lambda) = \frac{L}{2\pi} + \frac{1}{\pi} \sum_n \sum_{p \in P_n} \frac{l_p}{r_p} A_p \cos(\lambda l_p)$. Here L is the total length of all edges, P_n is the set of all periodic orbits of period n up to cyclic reordering (that is p_0, p_1, p_2 and p_1, p_2, p_0 are the same orbits), l_p is the length of the orbit, r_p its repetition number, and $A_p = \prod_{i=1}^n S_{p_i, p_{i+1}}$.

We will work on star graphs with $v + 1$ vertices (see figure 1): these are graphs with $V = \{0, \ldots, v\}$ and $E = \{(0, i), (i, 0), 1 \leqslant i \leqslant v\}$: v vertices are all connected to the center 0. The S-matrix elements are $S_{(0,i),(i,0)} = 1$ (this corresponds to trivial scattering), $S_{(i,0),(0,i)} = -1 + \frac{2}{v}$ (backscattering) and $S_{(i,0),(0,j)} = \frac{2}{v}$ (normal scattering). The lengths will be taken so that they are incommensurate, and that their distribution is peaked around 1: for instance, they can be chosen randomly, uniformly in $[1 - 1/2v, 1 + 1/2v]$, each length being independent from the other ones. With such a distribution, an orbit of period $2k$ has a length in $[2k - k/v, 2k + k/v]$; such intervals for different k's less than v do not overlap. The interesting limit will be the limit v that tends to infinity; in this limit, orbits with the biggest contribution A_p will be orbits with a large number of backscatterings.
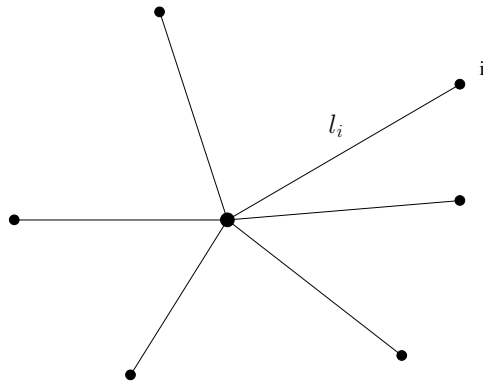
Figure 1. A star graph.

3. Two-point correlation function

The two-point correlation function is defined as $R_2(x) = \left(\frac{2\pi}{L}\right)^2 \langle d(\lambda)\, d(\lambda - \frac{2\pi x}{L}) \rangle$. The brackets denote a mean value with respect to the λ's, that is $\langle f \rangle = \lim_{\Lambda \to \infty} \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} f(\lambda)\, d\lambda$. Using the trace formula and performing the integral, one gets

$$R_2(x) = 1 + \frac{2}{L^2} \sum_{p,p'} \frac{l_p l_{p'}}{r_p r_{p'}} A_p A_{p'} \delta_{l_p - l_{p'}} \cos\left(\frac{2\pi x l_p}{L}\right),$$

where the sum is over the pairs of periodic orbits (p, p'), up to cyclic permutations, of lengths l_p and $l_{p'}$ and repetition numbers r_p and $r_{p'}$.

A combinatorial analysis of periodic orbits leads [8] to the formula

$$R_2(x) = 1 + \int_{-\infty}^{+\infty} K(\tau) \exp(-2i\pi x \tau)\, d\tau,$$

where K is given near $\tau = 0$ by

$$K(\tau) = \exp(-4\tau) + \sum_{j=2}^{\infty} \sum_{M=0}^{\infty} \frac{4^j}{j!} C_M \tau^{M+j+1}.$$

The C_M are defined by

$$C_M = (-2)^M \sum_{k_1 + \cdots + k_j + n_1 + \cdots + n_j = M} \frac{(K+j-1)!(N+j-1)!}{(M+j-1)!} \prod_{i=1}^{j} \frac{\binom{n_i+k_i}{n_i}}{(n_i+1)!(k_i+1)!},$$

with $K = \sum_{i=1}^{j} k_i$ and $N = \sum_{i=1}^{j} n_i$.

It is found to be different from the Poisson statistics, since $K(\tau)$ clearly depends on τ, and also different from random matrix statistics, since $K(0) = 1$ and not 0.

4. Three-point correlation function

The three-point correlation function is defined in a similar way as $\left(\frac{2\pi}{L}\right)^3 \langle d(\lambda)\, d(\lambda - \frac{2\pi x}{L})\, d(\lambda - \frac{2\pi y}{L}) \rangle |_\lambda$. Using the trace formula and developing, it is equal to $R_2(x) + R_2(y) + R_2(x-y) -$

$2 + R_3(x, y)$, where

$$R_3(x, y) = \left(\frac{2}{L}\right)^3 \lim_{\Lambda \to \infty} \frac{1}{2\Lambda} \int_{-\Lambda}^{\Lambda} \sum_{p,p',p''} \frac{l_p l_{p'} l_{p''}}{r_p r_{p'} r_{p''}} A_p A_{p'} A_{p''}$$

$$\times \cos(\lambda l_p) \cos\left(\left(\lambda - \frac{2\pi x}{L}\right) l_{p'}\right) \cos\left(\left(\lambda - \frac{2\pi y}{L}\right) l_{p''}\right) d\lambda.$$

The sum is over triplets of periodic orbits (p, p', p''), up to cyclic permutations, of lengths $l_p, l_{p'}, l_{p''}$ and repetition numbers $r_p, r_{p'}, r_{p''}$. Performing the integral, one gets

$$R_3(x, y) = \frac{2}{L^3} \sum_{p,p',p''} \frac{l_p l_{p'} l_{p''}}{r_p r_{p'} r_{p''}} A_p A_{p'} A_{p''} \left[\delta_{l_p + l_{p'} - l_{p''}} \cos\left(\frac{2\pi}{L}(y l_{p''} - x l_{p'})\right) \right.$$

$$\left. + \delta_{l_p - l_{p'} + l_{p''}} \cos\left(\frac{2\pi}{L}(y l_{p''} - x l_{p'})\right) + \delta_{l_p - l_{p'} - l_{p''}} \cos\left(\frac{2\pi}{L}(y l_{p''} + x l_{p'})\right) \right].$$

Let us have a look at the first term enforcing $l_{p''} = l_p + l_{p'}$. The other ones can obviously be treated in a similar way. We thus have to deal with

$$R_3^1 = \frac{2}{L^3} \sum_{\substack{p,p',p'' \\ l_{p''} = l_{p'} + l_p}} \frac{l_p l_{p'} (l_p + l_{p'})}{r_p r_{p'} r_{p''}} A_p A_{p'} A_{p''} \cos\left(\frac{2\pi}{L}(y l_p + (y - x) l_{p'})\right).$$

Now, since the edge lengths are incommensurate, the condition $l_{p''} = l_{p'} + l_p$ implies that the orbit p'' is formed of the union of the edges of orbit p and orbit p', which we will denote by $(p'') = (p) \cup (p')$. Moreover, since the edge lengths are sharply peaked around 1, the length of a periodic orbit of period $2k$ is nearly $2k$ and the total length L is nearly $2v$.

Thus one gets approximately

$$R_3^1 \simeq \frac{2}{v^3} \sum_{k,k'} kk'(k + k') \cos\left(\frac{4\pi}{L}(ky + k'(y - x))\right) \sum_{\substack{p \text{ period } 2k \\ p' \text{ period } 2k' \\ (p'')=(p)\cup(p')}} \frac{A_p A_{p'} A_{p''}}{r_p r_{p'} r_{p''}}.$$

Following [8], we will now take as a parameter the number j of distinct edges visited by the orbit p''.

4.1. The $j = 1$ case

The $j = 1$ case is a bit special—the orbits p, p' and p'' are here all formed of one and the same edge, for which there are $\binom{v}{1} = v$ choices; the repetition number of such an orbit of period $2k$ is then $r_p = k$, and the number of backscatterings is also k; thus, the contribution A_p is $\left(-1 + \frac{2}{v}\right)^k$. Hence, the $j = 1$ term is

$$R_3^{1,1} = \frac{2}{v^3} v \sum_{k,k'} \cos\left(\frac{2\pi}{v}(ky + k'(y - x))\right) \left(-1 + \frac{2}{v}\right)^{2k+2k'}.$$

In the $v \to \infty$ limit, putting $\tau = \frac{k}{v}$, the sum becomes an integral:

$$R_3^{1,1} \simeq 2 \iint_{\mathbb{R}^{+2}} d\tau \, d\tau' \, e^{-4\tau} e^{-4\tau'} \cos(2\pi(y\tau + (y - x)\tau')).$$

4.2. The $j = 2$ case

This case is still a bit special, because while p'' passes through two distinct edges, p and p' may still be restricted to one edge only. The orbit p'' of period $2k''$ is here formed by two different edges, denoted by 'a' and 'b', for which there are $\binom{v}{2}$ choices. It can be described as a succession of m'' packets of **a** and **b**. The number of scatterings and backscatterings of such an orbit is respectively $2m''$ (each change of packet contributes) and $k'' - 2m''$. Its decomposition in $p \cup p'$ will obviously also depend on the numbers n''_a of edges **a** and $n''_b = k'' - n''_a$ of edges **b** (we count the edges only when they depart from the root). One then needs to know how many different orbits p'' there are with given k'', m'', n''_a, n''_b, since they will all contribute the same. This number, as explained in [8], can be computed as follows: you first divide your **a**'s into m'' packets (order counts), then your **b**'s into m'' packets. Reminding that the number of partitions of an integer N into K parts is $\binom{N-1}{K-1}$, the number of ways to do this is $\binom{n''_a-1}{m''-1}\binom{n''_b-1}{m''-1}$. The orbits should be counted up to cyclic permutation, and the weight of each orbit has to be divided by its repetition number r. But one orbit with a given r corresponds to m''/r such decompositions. For example, the orbit **aabab** has $m = 2, r = 1$, and actually we get it twice, since it is the same up to cyclic permutation as **abaab**. **abab** has $m = 2$ and $r = 2$, and it is obtained only once. Hence each decomposition gets a $1/m''$ factor, as well as a contribution $A''_p = \left(-1 + \frac{2}{v}\right)^{k''-2m''}\left(\frac{2}{v}\right)^{2m''}$.

Each such orbit has then to be decomposed into two orbits p and p', composed of respectively n_a**a**'s and $n'_a = n''_a - n_a$**a**'s (and n_b and n'_b**b**'s), forming respectively m and m' packets. The period of p is $2(n_a + n_b)$. One has to pay attention to the fact that n_a can be 0, in which case $m = 1$ but the number of scatterings is then 0 and not 2 (this does not happen for $j > 1$ in the case of the two-point correlation function computed in [8] since there are then only two orbits p and p' visiting the same edges). This term has to be computed separately.

Putting all that together, the term $\tilde{R}_3^{1,2}$ where p, p' and p'' are all composed of two different edges is

$$\tilde{R}_3^{1,2} = \frac{2}{v^3}\binom{v}{2}\sum_{n''_a,n''_b}\sum_{\substack{n_a=1\\n'_a=n''_a-n_a}}^{n''_a-1}\sum_{\substack{n_b=1\\n'_b=n''_b-n_b}}^{n''_b-1}\sum_{m=1}^{\min(n_a,n_b)}\sum_{m'=1}^{\min(n'_a,n'_b)}\sum_{m''=1}^{\min(n''_a,n''_b)}(n''_a+n''_b)(n'_a+n'_b)(n_a+n_b)$$

$$\times \cos\left(\frac{2\pi}{v}((n_a+n_b)y+(y-x)(n'_a+n'_b))\right)\left(-1+\frac{2}{v}\right)^{2n''_a+2n''_b-2(m+m'+m'')}$$

$$\times \frac{\binom{n''_a-1}{m''-1}\binom{n''_b-1}{m''-1}}{m''}\frac{\binom{n'_a-1}{m'-1}\binom{n'_b-1}{m'-1}}{m'}\frac{\binom{n_a-1}{m-1}\binom{n_b-1}{m-1}}{m}\left(\frac{2}{v}\right)^{2(m+m'+m'')}.$$

One can now perform the $v \to \infty$ limit: denoting $q_i^* = \frac{n_i^*}{v}$ (the * is either void, ' or ''), sums over n's turn into integrals over q's, powers of $\left(1 - \frac{2}{v}\right)$ turn into exponentials of q's, terms such as $\frac{(n-1)!}{(n-m)!}$ turn into $(qv)^{m-1}$. Hence, writing $q' = q'' - q$:

$$\tilde{R}_3^{1,2} \simeq \int_{(\mathbb{R}^+)^4}dq_a\,dq_b\,dq''_a\,dq''_b(q_a+q_b)(q''_a+q''_b)(q'_a+q'_b)$$

$$\times \cos(2\pi((q_a+q_b)y+(y-x)(q'_a+q'_b)))$$

$$\times \exp(-4(q''_a+q''_b))4^3\sum_{m''\geqslant 1}\frac{(4q''_a q''_b)^{m''-1}}{(m''-1)!m''!}\sum_{m'\geqslant 1}\frac{(4q'_a q'_b)^{m'-1}}{(m'-1)!m'!}\sum_{m\geqslant 1}\frac{(4q_a q_b)^{m-1}}{(m-1)!m!}$$

where the integral is over $\{q''_{a,b} \geqslant 0$ and $0 \leqslant q_{a,b} \leqslant q''_{a,b}\}$. Using the modified Bessel function $I_1(z) = (z/2) \sum_{k \in \mathbb{N}} \frac{(z^2/4)^k}{k!(k+1)!}$, and denoting $\mathfrak{I}(x) = I_1(4\sqrt{x})/\sqrt{x}$, one gets

$$\tilde{R}_3^{1,2} \simeq 8 \iint_{(\mathbb{R}^+)^2} d\tau \, d\tau' \, e^{-4\tau} \, e^{-4\tau'} \cos(2\pi(y\tau + (y-x)\tau')\tau\tau'(\tau + \tau')$$

$$\times \int_0^\tau dq \int_0^{\tau'} dq' \mathfrak{I}((q+q')(\tau + \tau' - q - q'))\mathfrak{I}(q(\tau - q))\mathfrak{I}(q'(\tau' - q')).$$

Let us now look at the case where one of the orbits (p, p') is composed of only one edge; for example, the contribution of the term $n_a = 0$ (and thus $n'_a = n''_a$) is

$$\frac{2}{v^3} \binom{v}{2} \sum_{n''_a, n''_b} \sum_{m''=1}^{\min(n''_a, n''_b)} \sum_{\substack{n_b=1 \\ n'_b = n''_b - n_b}}^{n''_b - 1} \sum_{m'=1}^{\min(n''_a, n'_b)} (n''_a + n''_b)(n''_a + n'_b) \frac{\binom{n''_a - 1}{m'' - 1}\binom{n''_b - 1}{m'' - 1}}{m''} \frac{\binom{n''_a - 1}{m' - 1}\binom{n'_b - 1}{m' - 1}}{m'}$$

$$\times \cos\left(\frac{2\pi}{v}(n_b y + (y - x)(n''_a + n'_b))\right)\left(-1 + \frac{2}{v}\right)^{2(n''_a + n''_b - m' - m'')}\left(\frac{2}{v}\right)^{2(m' + m'')}.$$

Since there are two symbols **a** and **b**, and since there are two orbits p and p' that can be degenerate, the total contribution $\bar{R}_3^{1,2}$ of the 'one degenerate orbit' case is, when v tends to infinity,

$$\bar{R}_3^{1,2} \simeq 8 \iint_{(\mathbb{R}^+)^2} d\tau \, d\tau' \, e^{-4\tau} \, e^{-4\tau'} \cos(2\pi(y\tau + (y - x)\tau'))(\tau + \tau')$$

$$\times \left[\tau' \int_0^{\tau'} dq \, \mathfrak{I}(q(\tau + \tau' - q))\mathfrak{I}(q(\tau' - q)) + \tau \int_0^\tau dq \, \mathfrak{I}(q(\tau + \tau' - q))\mathfrak{I}(q(\tau - q))\right].$$

When both orbits p and p' consist of one edge (this problem is specific to the $j = 2$ case), the contribution $\hat{R}_3^{1,2}$ is

$$\hat{R}_3^{1,2} = \frac{2}{v^3} \binom{v}{2} \sum_{n''_a, n''_b} \sum_{m''=1}^{\min(n''_a, n''_b)} n''_a n''_b (n''_a + n''_b) \cos\left(\frac{2\pi}{v}(n''_b y + n''_a(y - x))\right)$$

$$\times \frac{\binom{n''_a - 1}{m'' - 1}\binom{n''_b - 1}{m'' - 1}}{m''} \frac{1}{n''_a n''_b}\left(-1 + \frac{2}{v}\right)^{2n''_a + 2n''_b - 2m''}\left(\frac{2}{v}\right)^{2m''}$$

$$\simeq \iint_{(\mathbb{R}^+)^2} d\tau \, d\tau' \, e^{-4\tau} \, e^{-4\tau'} \cos(2\pi(y\tau + (y - x)\tau')(\tau + \tau')\mathfrak{I}(\tau\tau').$$

All in all, this gives

$$R_3^{1,2} = \tilde{R}_3^{1,2} + \bar{R}_3^{1,2} + \hat{R}_3^{1,2} = \iint_{(\mathbb{R}^+)^2} d\tau \, d\tau' \, e^{-4\tau} \, e^{-4\tau'} \cos(2\pi(y\tau + (y - x)\tau')(\tau + \tau')$$

$$\times \left[\mathfrak{I}(\tau\tau') + 8\tau\tau' \int_0^\tau dq \int_0^{\tau'} dq' \mathfrak{I}((q+q')(\tau + \tau' - q - q'))\mathfrak{I}(q(\tau - q))\mathfrak{I}(q'(\tau' - q'))\right.$$

$$\left. + 8\tau' \int_0^{\tau'} dq \, \mathfrak{I}(q(\tau + \tau' - q))\mathfrak{I}(q(\tau' - q)) + 8\tau \int_0^\tau dq \, \mathfrak{I}(q(\tau + \tau' - q))\mathfrak{I}(q(\tau - q))\right].$$

4.3. The $j > 2$ case

The orbit p'' is now formed by j edges, denoted by $(\mathbf{1}, \mathbf{2}, \ldots \mathbf{j})$ for which there are $\binom{v}{j}$ choices. We will denote by n''_i the number of edges $\mathbf{i}$ in the orbit p'', by m''_i the number of

groups of adjacent $\mathbf{i}$, and by $\mathbf{n}''$ and $\mathbf{m}''$ the corresponding vectors of $\mathbb{Z}^j$. For example, the orbit $p'' = \mathbf{11212332}$ has $j = 3$, $\mathbf{n}'' = (3, 3, 2)$ and $\mathbf{m}'' = (2, 3, 1)$. The period of the orbit is $2 \sum_{i=1}^{j} n_i'' = 2N''$, the number of scatterings is the number of groups $M'' = \sum_{i=1}^{j} m_j''$ and the number of backscatterings is $\sum_{i=1}^{j}(n_j'' - m_j'') = N'' - M''$. We will denote by $Q_{\mathbf{n}''}^{\mathbf{m}''}$ the number of orbits with given j, $\mathbf{n}''$ and $\mathbf{m}''$, each weighted by $1/r_{p''}$.

Such an orbit has to be decomposed into two orbits p and p', consisting respectively of n_i and $n_i' = n_i'' - n_i$ edges $\mathbf{i}$, and of m_i and m_i' groups of $\mathbf{i}$. Some n_i or n_i' can be zero, but not all of them, and as in the $j = 2$ case, we will have to consider separately the case where all the n_i are zero but one.

To lighten the formulae, we will denote by $\prod^* T^* = TT'T''$ (where T can be any quantity we have defined for p, p' and p''); $\vec{1} \stackrel{\text{def}}{=} (1, 1, \dots, 1)$; if u and v are vectors, $U \stackrel{\text{def}}{=} \sum_{i=1}^{j} u_i$, $u^v \stackrel{\text{def}}{=} \prod_{i=1}^{j} u_i^{v_i}$, $u! \stackrel{\text{def}}{=} \prod_{i=1}^{j}(u_i)!$ and $u \leqslant v$ means $u_i \leqslant v_i$ for all i.

In the general case where all orbits consist of at least two different edges, we have

$$\tilde{R}_3^{1,j} = \frac{2}{v^3} \binom{v}{j} \sum{}' NN''(N'' - N) \cos\left(\frac{2\pi}{v}(N''y - x(N'' - N))\right)$$
$$\times \prod^* Q_{\mathbf{n}^*}^{\mathbf{m}^*} \prod^* \left(-1 + \frac{2}{v}\right)^{k^* - M^*} \left(\frac{2}{v}\right)^{M^*}.$$

Here $\sum'$ denotes a sum over the vectors $\mathbf{n}^*$ and $\mathbf{m}^*$ satisfying

$$\begin{cases} \vec{0} \leqslant \mathbf{n} \leqslant \mathbf{n}'' \\ \mathbf{n}' = \mathbf{n}'' - \mathbf{n} \\ \vec{1} \leqslant \mathbf{m}^* \leqslant \mathbf{n}^*. \end{cases}$$

(Rigorously, we should avoid the case where $\mathbf{n}$ or $\mathbf{n}'$ is $\mathbf{0}$, but thanks to the $\frac{1}{v^3}$ term, its contribution will disappear when the sums turn into integrals in the $v \to \infty$ limit.)

All we need now is to determine the numbers $Q_{\mathbf{n}^*}^{\mathbf{m}^*}$. The computation is done in [8], let us just present the ideas behind. We will count the sequences of $(\mathbf{1}, \mathbf{2}, \dots, \mathbf{j})$ such that there are $n_i\mathbf{i}$'s and m_i groups of $\mathbf{i}$, starting by a group of $\mathbf{1}$, and not ending by a group of $\mathbf{1}$. Due to cyclic permutations, this is not exactly the same as counting periodic orbits, but nearly; for example, the orbit $\mathbf{11212332}$ corresponds to the $n_1/r = 2$ sequences $\mathbf{11212332}$ and $\mathbf{12332112}$, whereas the orbit $\mathbf{23112311}$ corresponds to the only $n_1/r = 1$ sequence $\mathbf{11231123}$. Q is then exactly the number of such sequences divided by n_1. To compute this number, one counts the number of ways to put the $n_i\mathbf{i}$'s in m_i packets, and then to arrange such packets, starting by $\mathbf{1}$, keeping the order of the groups of a given symbol, not ending with $\mathbf{1}$, and in such a way that two groups of the same symbol are not neighbors. The first step gives a factor $\prod_{i=1}^{j} \binom{n_i - 1}{m_i - 1}$. The second step is the most tricky one; it can be evaluated using an exclusion/inclusion principle. All in all, this gives

$$Q_{\mathbf{n}}^{\mathbf{m}} = \prod_{i=1}^{j} \binom{n_i - 1}{m_i - 1} (-1)^M \sum_{\vec{1} \leqslant \mathbf{t} \leqslant \mathbf{m}} \frac{(-1)^T}{T} \binom{T}{t_1, \dots, t_j} \prod_{i=1}^{j} \binom{m_i - 1}{t_i - 1}.$$

Now all one has to do is to perform the $v \to \infty$ limit. Using $\sum_{m=1}^{\infty} \sum_{t=1}^{m} = \sum_{t=1}^{\infty} \sum_{m=t}^{\infty}$ and $\sum_{m=t}^{\infty} \frac{x^{m-1}}{(m-t)!} = x^{t-1} \exp x$ and introducing $\tau = \sum_{i=1}^{j} q_j$, one gets

$$\tilde{R}_3^{1,j} \simeq \frac{2}{j!} \sum_{\mathbf{t}, \mathbf{t}', \mathbf{t}''} \int_{\mathbf{q}'' \geqslant \vec{0}} \prod_{i=1}^{j} \mathrm{d}q_j'' \int_{\mathbf{0} \leqslant \mathbf{q} \leqslant \mathbf{q}''} \prod_{i=1}^{j} \mathrm{d}q_j \, \tau \tau''(\tau'' - \tau) \cos(2\pi(\tau x + \tau''(y - x)))$$
$$\times \prod^* (\mathbf{q}^*)^{\mathbf{t}^* - \mathbf{1}} \frac{(T-1)!(T'-1)!(T''-1)!}{\prod^* \mathbf{t}^*!(\mathbf{t}^* - \mathbf{1})!} (-2)^{T + T' + T''}.$$

Using the identity $\int_{q_i \geqslant 0, \sum_{i=1}^{j} q_i = \tau} dq_1 \cdots dq_{j-1} \prod_{i=1}^{j} q_i^{m_i} = \frac{\prod_{i=1}^{j} m_i!}{(M+j-1)!} \tau^{M+j-1}$ and developing $(\mathbf{q''})^{\mathbf{t''}-\mathbf{1}} = (\mathbf{q} + \mathbf{q'})^{\mathbf{t''}-\mathbf{1}}$, this becomes

$$\tilde{R}_3^{1,j} \simeq \frac{2}{j!} \iint_{\mathbb{R}^{+2}} d\tau \, d\tau' (\tau + \tau') \cos(2\pi(y\tau + (y-x)\tau')) \sum_{\mathbf{t} \geqslant \vec{1}} \sum_{\mathbf{t'} \geqslant \vec{1}} \sum_{\mathbf{t''} \geqslant \vec{1}} \sum_{\vec{0} \leqslant \mathbf{s} < \mathbf{t''}}$$

$$\times \prod_{i=1}^{j} \frac{\binom{s_i + t_i - 1}{s_i}\binom{t_i'' + t_i' - s_i - 2}{t_i' - 1}}{(t_i)!(t_i')!(t_i'')!} \frac{(T-1)!(T'-1)!(T''-1)!}{(S+T-1)!(T''+T'-S-j-1)!}$$

$$\times (-2)^{T+T'+T''} \tau^{S+T}(\tau')^{T'+T''-S-j}.$$

The case where p consists of only one edge gives a factor

$$\frac{2j}{v^3} \binom{v}{j} \sum^2 N''(N'' - N) \cos\left(\frac{2\pi}{v}(N''y - x(N'' - N))\right)$$

$$\times Q_{\mathbf{n''}}^{\mathbf{m''}} Q_{\mathbf{n'}}^{\mathbf{m'}} \left(-1 + \frac{2}{v}\right)^{2N'' - M' - M''} \left(\frac{2}{v}\right)^{M'' + M'}$$

where $\sum^2$ denotes a sum over the vectors $\mathbf{n''}$, $\mathbf{m''}$, $\mathbf{m'}$ and over the integer n_1 satisfying

$$\begin{cases} 1 \leqslant n_1 \leqslant n_1'' \\ \vec{1} \leqslant \mathbf{m''} \leqslant \mathbf{n''} \\ \vec{1} \leqslant \mathbf{m'} \leqslant \mathbf{n''} \\ 1 \leqslant m_1' \leqslant n_1'' - n_1. \end{cases}$$

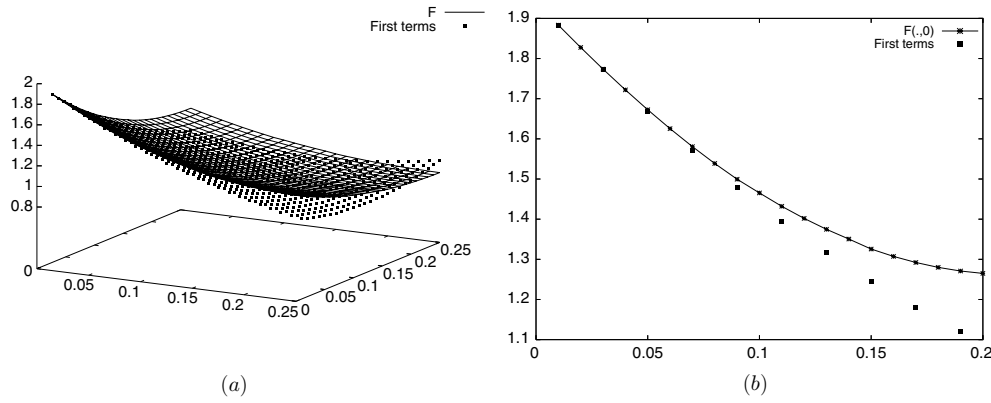Hence, in the limit $v \to \infty$, the contribution of p or p' consisting of one edge only is

$$\bar{R}_3^{1,j} \simeq \frac{2}{(j-1)!} \iint_{\mathbb{R}^{+2}} d\tau \, d\tau' (\tau + \tau') \cos(2\pi(y\tau + (y-x)\tau'))$$

$$\times \sum_{t_i', t_i''} \sum_{s=0}^{t_1''-1} \frac{\binom{s+t_1'-1}{s} \prod_{i=2}^{j}\binom{t_i''+t_i'-2}{t_i'-1}}{(t_1''-1-s)! \prod_{i=1}^{j}(t_i')! \prod_{i=1}^{j}(t_i'')!} \frac{(T'-1)!(T''-1)!}{(T''+T'-t_1''+s-j)!}(-2)^{T'+T''}$$

$$\times [(\tau')^{T''+T'-t_1''+s-j+1}\tau^{t_1''-1-s}\exp(-2\tau) + \tau^{T''+T'-t_1''+s-j+1}(\tau')^{t_1''-1-s}\exp(-2\tau')].$$

Finally, the contribution for $j > 2$ is $R_3^{1,j} = \tilde{R}_3^{1,j} + \bar{R}_3^{1,j}$.

4.4. The three-point correlation function

Putting everything back together, the three-point correlation function can be written as $R_2(x) + R_2(y) + R_2(x-y) - 2 + \iint_{\mathbb{R}^{+2}} d\tau \, d\tau'(\cos(2\pi(y\tau + (y-x)\tau') + \cos(2\pi(y\tau' - x(\tau + \tau')) + \cos(2\pi(y\tau + x\tau'))F(\tau, \tau')$, where $F = F_1 + F_2 + F_3 + F_4$ is given by $F_1(\tau, \tau') = 2e^{-4\tau} e^{-4\tau'}$,

$$F_2(\tau, \tau') = e^{-4\tau} e^{-4\tau'}(\tau + \tau')\left[\Im(\tau\tau') + 8\tau' \int_0^{\tau'} dq \, \Im(q(\tau + \tau' - q))\Im(q(\tau' - q))\right.$$

$$+ 8\tau \int_0^{\tau} dq \, \Im(q(\tau + \tau' - q))\Im(q(\tau - q))$$

$$\left. + 8\tau\tau' \int_0^{\tau} dq \int_0^{\tau'} dq' \Im((q + q')(\tau + \tau' - q - q'))\Im(q(\tau - q))\Im(q'(\tau' - q'))\right]$$

Figure 2. (*a*) The function F (line) and its expansion up to order 2 (squares); (*b*) corresponds to $\tau' = 0$.

$$F_3(\tau, \tau') = \sum_{j \geqslant 3} \frac{2}{j!} (\tau + \tau') \sum_{\mathbf{t} \geqslant \bar{1}} \sum_{\mathbf{t}' \geqslant \bar{1}} \sum_{\mathbf{t}'' \geqslant \bar{1}} \sum_{0 \leqslant \mathbf{s} < \mathbf{t}''} \prod_{i=1}^{j} \frac{\binom{s_i + t_i - 1}{s_i}\binom{t_i'' + t_i' - s_i - 2}{t_i' - 1}}{(t_i)!(t_i')!(t_i'')!}$$
$$\times \frac{(T-1)!(T'-1)!(T''-1)!}{(S+T-1)!(T''+T'-S-j-1)!} (-2)^{T+T'+T''} \tau^{S+T} (\tau')^{T'+T''-S-j}$$

$$F_4(\tau, \tau') = \sum_{j \geqslant 3} \frac{2}{(j-1)!} (\tau + \tau') \sum_{t_i', t_i''} \sum_{s=0}^{t_1''-1} \frac{\binom{s+t_i'-1}{s} \prod_{i=2}^{j} \binom{t_i''+t_i'-2}{t_i'-1}}{(t_i''-1-s)! \prod_{i=1}^{j} (t_i')! \prod_{i=1}^{j} (t_i'')!}$$
$$\times \frac{(T'-1)!(T''-1)!}{(T''+T'-t_1''+s-j)!} (-2)^{T'+T''} [(\tau')^{T''+T'-t_1''+s-j+1} \tau^{t_1''-1-s} \exp(-2\tau)$$
$$+ \tau^{T''+T'-t_1''+s-j+1} (\tau')^{t_1''-1-s} \exp(-2\tau')].$$

4.5. Small τ, τ' expansion

The first contribution for the 'general' jth term where no orbit consists of only one edge is $(\tau\tau')^j(\tau + \tau')$, and the first contribution when one orbit is degenerate is $(\tau')^j(\tau + \tau')$. Keeping only the first terms, and using the expansion of $\Im(x) = 2 + 4x + O(x^2)$ when x is small, one thus gets $F(\tau, \tau') = 2 - 6\tau - 6\tau' + 16\tau\tau' + 8\tau^2 + 8(\tau')^2 + o(\tau^2, \tau'^2, \tau\tau')$. Figure 2 presents a graph of the function F, together with the first terms of its expansion.

5. Conclusion and perspectives

We have obtained here a formula for the three-point correlation function of star graphs, allowing in principle to get its small τ, τ' expansion at any fixed order s by keeping terms up to $j = s - 1$. One should be able to get the first order of the expansion for the n-point correlation function in the same way by computing the $j = 1$ and $j = 2$ contributions. Since the statistics of the eigenvalues are characterized by all the n-point correlation functions, this is a small step toward knowing a bit more of this intermediate statistics for star graphs.

This specific result for the three-point function could also be of some help in computing the form factor of two star graphs S_1 and S_2 glued together by an edge **e** linking their centers.

Indeed, the form factor is a sum over pairs of orbits (p, q) visiting the same edges. Let us consider all the couples (p, q) corresponding to a given set of edges. To get the main term in the small τ expansion, it would seem reasonable to group the edges corresponding to each graph together, and thus to consider first the couples (p, q) such that p and q can be written as $p = p_1 E p_2 E$ and $q = q_1 E q_2 E$, where p_i and q_i are orbits on S_i, and E denotes any sequence $\mathbf{e} \cdots \mathbf{e}$ of edges $\mathbf{e}$. The sum over such p and q should thus involve a sum over (p_1, q_1) and a sum over (p_2, q_2), and the product of the form factors of each star graph should appear here. The next term would correspond to decompositions $p = p_1 E p_2 E p_1' E p_2'$ and $q = q_1 E q_2 E$. The sum over orbits p, q would then involve a sum on each star graph over orbits p_i, p_i', q_i such that $(q_i) = (p_i) \cup (p_i')$, similar to the sum computed in the three-point correlation function. Of course, the exact calculation would involve all the n-point correlation functions as well as combinatorial factors to insert the E's, but the first terms of the expansion should already give an insight of what happens.

Acknowledgments

References

[1] Kottos T and Smilansky U 1999 Periodic orbit theory and spectral statistics for quantum graphs *Ann. Phys.* **274** 76–124

[2] Gnutzmann S and Smilansky U 2006 Quantum graphs: applications to quantum chaos and universal spectral statistics *Adv. Phys.* **55** 527–625

[3] Barra F and Gaspard P 2000 On the level spacing distribution in quantum graphs *J. Stat. Phys.* **101** 283–319

[4] Kostrykin V, Potthoff J and Schrader R 2007 Heat kernels on metric graphs and a trace formula *Preprint* math-ph/0701009

[5] Kottos T and Smilansky U 2003 Quantum graphs: a simple model for chaotic scattering *J. Phys. A: Math. Gen.* **36** 3501–24

[6] Gnutzmann S and Altland A 2004 Universal spectral statistics in quantum graphs *Phys. Rev. Lett.* **93**

[7] Berkolaiko G 2000 Quantum star graphs and related systems *PhD Thesis* University of Bristol

[8] Berkolaiko G and Keating J P 1999 Two point spectral correlations for star graphs *J. Phys. A: Math. Gen.* **32** 7827–41

[9] Winn B 2006 On the trace formula for quantum star graphs *Quantum Graphs and Their Application* vol 415 ed G Berkolaiko, R Carlson, S A Fulling and P Kuchment (Providence, RI: American Mathematical Society) pp 293–307

[10] Berkolaiko G, Bogomolny E B and Keating J P 2001 Star graphs and Seba billiards *J. Phys. A: Math. Gen.* **34** 335–50

[11] Kopp M, Schanz H and Geisel T 2005 The spectral form factor of double star graphs. Poster